MAX PLANCK INSTITUTE
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CSC
COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

`emgr` - EMpirical GRamian Framework (Version 5.4)

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`emgr` for (Nonlinear) Input-Output Systems

In system theory and control engineering, the system Gramian matrices of linear input-output systems encoding controllability, observability and minimality, have wide-spread use, for example in: Model reduction or uncertainty quantification. Empirical Gramian matrices correspond to the linear system Gramians, but extend to parametric and nonlinear systems due to their data-driven computation.

The empirical Gramian framework - `emgr` - is an open-source toolbox, compatible with MathWorks MATLAB and GNU Octave, which enables the computation of various empirical system Gramians.

Features

- Modular: Interfaces for Solver, inner product kernels & distributed memory computing.
- Configurable: Algorithmic variants selectable via flags.
- Universal: Non-Symmetric option for all cross Gramians.
- Fast: Vectorized and (`parfor`) parallelizable.
- Free: Open-source licensed.
- Modern: Functional paradigm design.
- Compact: Less than 500 lines of code in a single file.

Empirical Gramians

- Empirical controllability Gramian W_C
- Empirical observability Gramian W_O
- Empirical cross Gramian W_X
- Empirical linear cross Gramian W_Y (accelerated variant for linear systems)
- Empirical sensitivity Gramian W_S (controllability of state and parameters)
- Empirical identifiability Gramian W_I (observability of state and parameters)
- Empirical joint Gramian W_J (observability of parameters and minimality of state)

Input-Output Systems

(Possibly Nonlinear) Input-Output System:

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), u(t), \theta) \\ y(t) &= g(t, x(t), u(t), \theta) \end{aligned}$$

- Input: $u : \mathbb{R} \to \mathbb{R}^M$
- State: $x : \mathbb{R} \to \mathbb{R}^N$
- Output: $y : \mathbb{R} \to \mathbb{R}^Q$
- Parameter: $\theta \in \mathbb{R}^P$
- Vector Field: $f : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^M \times \mathbb{R}^P \to \mathbb{R}^N$
- Output Functional: $g : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^M \times \mathbb{R}^P \to \mathbb{R}^Q$

Model Order Reduction

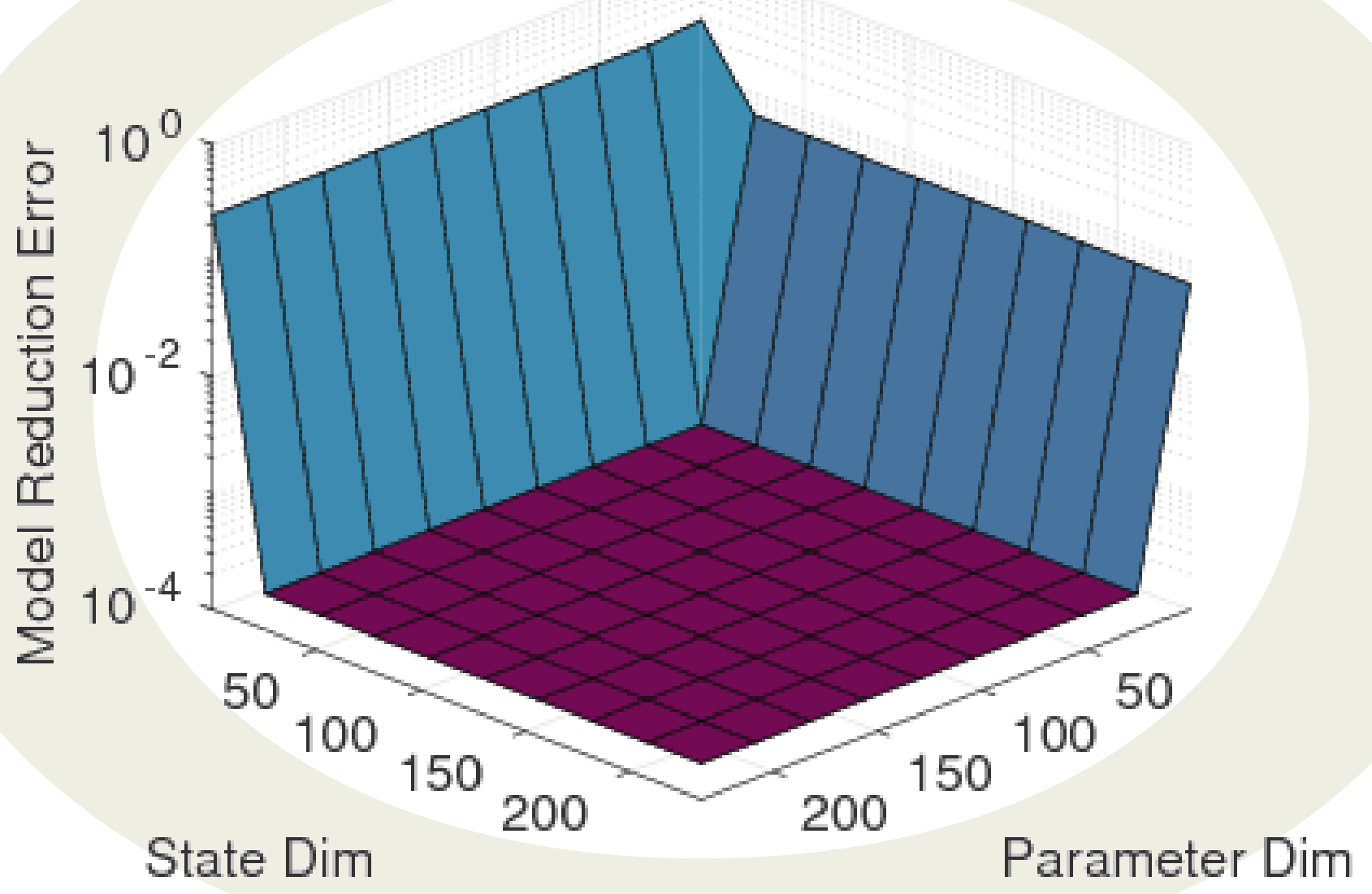
How to obtain a reduced order model that preserves the input-output behavior of the full order model?

State-Space Reduction:

1. Compute **empirical Gramians**:
    - Empirical controllability Gramian (Proper Orthogonal Decomposition),
    - Empirical controllability and observability Gramian (Balanced Truncation),
    - Empirical cross Gramian (Approximate Balancing / Direct Truncation).
2. Determine sorting projections.
3. Truncate projections.
4. Apply truncated projections:

$$\begin{aligned} \dot{x}_r(t) &= V^\intercal f(t, \bar{x} + U x_r(t), u(t), \theta) \\ \tilde{y}(t) &= \quad g(t, \bar{x} + U x_r(t), u(t), \theta) \end{aligned}$$

- Steady-State: $\bar{x} \in \mathbb{R}^N$
- Reduced State Dim.: $n \ll N$
- Projections: $U, V \in \mathbb{R}^{N \times n}$, $V^\intercal U = \mathbb{1}$
- Reduced State: $x_r(t) := V^\intercal (x(t) - \bar{x})$



Reducibility of a nonlinear hyperbolic network model.

Combined State and Parameter Reduction:

1. Compute **empirical Gramians**:
    - Empirical sensitivity and observability Gramian (Controllability-based),
    - Empirical controllability and identifiability Gramian (Observability-based),
    - Empirical joint Gramian (Minimality-based).
2. Determine state and parameter projection.
3. Truncate projections.
4. Apply truncated projections:

$$\begin{aligned} \dot{x}_r(t) &= V^\intercal f(t, \bar{x} + U x_r(t), u(t), \bar{\theta} + \Pi \theta_r) \\ \tilde{y}(t) &= \quad g(t, \bar{x} + U x_r(t), u(t), \bar{\theta} + \Pi \theta_r) \end{aligned}$$

- Nominal Parameter: $\bar{\theta} \in \mathbb{R}^P$
- Reduced Param. Dim.: $p \ll P$
- Projections: $\Pi, \Lambda \in \mathbb{R}^{P \times p}$, $\Lambda^\intercal \Pi = \mathbb{1}$
- Reduced Parameter: $\theta_r := \Lambda^\intercal (\theta - \bar{\theta})$

System Indices

How to quantify system properties with Gramians?

- Hankel Singular Values
- System Gain
- System Entropy
- System Symmetry
- Nyquist Enclosed Area
- Robustness Index
- Cauchy Index
- Energy Fraction
- State Index
- Ellipsoid Volume
- System Frobenius-Norm
- H_2-Norm
- Hankel-Norm Lower Bound
- H_∞-Norm Upper Bound
- L_1-Norm Upper Bound
- Fault Recoverability Index

`http://gramian.de`

Nonlinearity Quantification

How nonlinear is the system?

1. Compute trace of **empirical Gramian**:
    - Empirical controllability Gramian (input nonlinearity),
    - Empirical cross Gramian (state nonlinearity),
    - Empirical observability Gramian (output nonlinearity).
2. Compute traces of linearized system Gramian.
3. Difference in traces exposes nonlinearity.

Nonlinearity Measures:

$$\begin{aligned} N_c &= |\operatorname{tr}(W_C) - \operatorname{tr}(W_{C,\text{lin}})| \\ N_x &= |\operatorname{tr}(W_X) - \operatorname{tr}(W_{X,\text{lin}})| \\ N_o &= |\operatorname{tr}(W_O) - \operatorname{tr}(W_{O,\text{lin}})| \end{aligned}$$

Decentralized Control

Which inputs affect which outputs?

1. Compute **Gramian** for each SISO subsystem of a MIMO:
    - Empirical cross Gramian,
    - Empirical controllability and observability Gramian.
2. Traces of subsystem Gramians yield participation matrix.
3. Row or column maximum indicate principal SISOs.

Participation Matrix:

$$P = \begin{pmatrix} |\operatorname{tr}(W_{X,11})| & \dots & |\operatorname{tr}(W_{X,1Q})| \\ \vdots & \ddots & \vdots \\ |\operatorname{tr}(W_{X,M1})| & \dots & |\operatorname{tr}(W_{X,MQ})| \end{pmatrix}$$

Sensitivity Analysis

Which parameters affect system dynamics?

1. Treat parameters as system inputs.
2. Compute **empirical Gramians** for each parameter input:
    - Empirical sensitivity Gramian,
    - Empirical cross Gramian.
3. Traces of Gramians reveal sensitivities.

Parameter Sensitivities:

$$S(\theta_i) = |\operatorname{tr}(W_{S,i})|$$

Parameter Identification

What combination of parameters affects system dynamics?

1. Treat parameters as system states.
2. Compute **empirical Gramian**:
    - Empirical identifiability Gramian,
    - Empirical joint Gramian.
3. An SVD of parameter Gramian yields transformation.

Parameter Identifiability:

$$I(\theta_i) = \sigma_{\operatorname{argmax}_j(\Pi_{ij})}(W_I)$$

`README`

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