Reducibility of a nonlinear hyperbolic network model.

Decentralized Control

DC

MOR

Model Order Reduction

The empirical Gramian framework - emgr - is an open-source toolbox, compatible with MathWorks MATLAB and GNU Octave, which enables the computation of various empirical system Gramians.

Empirical Gramians

- Empirical controllability Gramian \( W_c \)
- Empirical observability Gramian \( W_o \)
- Empirical cross Gramian \( W_{xy} \)
- Empirical sensitivity Gramian \( W_s \)
- Empirical identifiability Gramian \( W_{id} \)
- Empirical joint Gramian \( W_j \)

Input-Output Systems

(Possibly Nonlinear) Input-Output System:

\[
\begin{align*}
  k(t) &= f(t, x(t), u(t), u_0(t), i) \\
  y(t) &= g(t, x(t), u(t), i).
\end{align*}
\]

Input: \( u : \mathbb{R} \rightarrow \mathbb{R}^m \)
State: \( x : \mathbb{R} \rightarrow \mathbb{R}^n \)
Output: \( y : \mathbb{R} \rightarrow \mathbb{R}^p \)

Parameter: \( \theta \in \mathbb{R}^d \)
Vector Field: \( f \in \mathbb{R}^n \rightarrow \mathbb{R}^m \)
Output Functional: \( g \in \mathbb{R}^p \rightarrow \mathbb{R}^m \)

Model Order Reduction

How to obtain a reduced order model that preserves the input-output behavior of the full order model?

1. Compute empirical Gramians:
   - Empirical controllability Gramian (Proper Orthogonal Decomposition).
   - Empirical controllability and observability Gramian (Balanced Truncation).
   - Empirical cross Gramian (Approximated Balancing / Direct Truncation).
2. Determine sorting projections.
3. Truncate projections.
4. Apply truncated projections:
   - Steady-State: \( x \in \mathbb{R}^n \)
   - Reduced State Dim.: \( n < N \)
   - Projections: \( U, V \in \mathbb{R}^{N \times N}, V U = 1 \)

Combined State and Parameter Reduction:

1. Compute empirical Gramians:
   - Empirical sensitivity and observability Gramian (Controllability-based).
   - Empirical controllability and identifiability Gramian (Observability-based).
   - Empirical joint Gramian (Minimality-based).
2. Determine state and parameter projection.
3. Truncate projections.
4. Apply truncated projections:
   - Nominal Parameter: \( \theta \in \mathbb{R}^d \)
   - Reduced State Dim.: \( p < P \)
   - Projections: \( \Pi, \Lambda \in \mathbb{R}^{P \times P}, \Pi \Lambda = 1 \)

System Indices

- Hankel Singular Values
- System Gain
- System Entropy
- System Symmetry
- Nyquist Enclosed Area
- Robustness Index
- Cauchy Index
- Energy Fraction
- State Index
- Ellipsoid Volume
- System Frobenius-Norm
- \( H_\infty \) Norm
- Hankel Norm Lower Bound
- \( H_\infty \) Norm Upper Bound
- \( L_\infty \) Norm Upper Bound
- Fault Recoverability Index

Decentralized Control

Sensitivity Analysis

Which parameters affect which outputs?

1. Compute Gramian for each SISO subsystem of a MIMO:
   - Empirical cross Gramian.
   - Empirical controllability and observability Gramian.
2. Traces of subsystem Gramians yield participation matrix.
3. Row or column maximum indicate principal SISOs.

Participation Matrix:

\[
P = \begin{pmatrix}
  |\text{tr}(W_{x11})| & ... & |\text{tr}(W_{x1n})| \\
  |\text{tr}(W_{x21})| & ... & |\text{tr}(W_{x2n})| \\
  ... & ... & ... \\
  |\text{tr}(W_{xn1})| & ... & |\text{tr}(W_{xnn})|
\end{pmatrix}
\]

Which parameters affect system dynamics?

1. Treat parameters as system inputs.
2. Compute empirical Gramians for each parameter input:
   - Empirical sensitivity Gramian.
   - Empirical cross Gramian.
3. Traces of Gramians reveal sensitivities.

Parameter Sensitivities:

\[ S(k) = |\text{tr}(W_{ki})| \]

Nonlinearity Quantification

How nonlinear is the system?

1. Compute trace of empirical Gramian:
   - Empirical controllability Gramian (input nonlinearity).
   - Empirical cross Gramian (state nonlinearity).
   - Empirical observability Gramian (output nonlinearity).
2. Compute traces of linearized system Gramian.
3. Difference in traces exposes nonlinearity.

Nonlinearity Measures:

\[
N_c = |\text{tr}(W_{x11})| - |\text{tr}(W_{x1n})| \\
N_o = |\text{tr}(W_{11})| - |\text{tr}(W_{1n})| \\
N_p = |\text{tr}(W_{x11})| - |\text{tr}(W_{x1n})|
\]

Parameter Identification

What combination of parameters affects system dynamics?

1. Treat parameters as system states.
2. Compute empirical Gramian.
   - Empirical identifiability Gramian.
   - Empirical joint Gramian.
3. An SVD of parameter Gramian yields transformation.

Parameter Identifiability:

\[ S(k) = |\text{svd}(\Pi^T W_{pi})| \]

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